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Globally maximal arithmetic groups ☆

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1. Introduction

Let G be a linear algebraic group defined over $\mathbb{Q}$, and assume that $G(\mathbb{R})$ is compact. Let $\widehat{\mathbb{Q}} := \widehat{\mathbb{Z}} \otimes \mathbb{Q}$ be the ring of finite adèles (see, e.g., [20]). Every arithmetic subgroup Γ of $G(\mathbb{Q})$ is finite, and is obtained by choosing an open, compact subgroup K of $G(\widehat{\mathbb{Q}})$ and defining $\Gamma = K \cap G(\mathbb{Q})$ in $G(\widehat{\mathbb{Q}})$. We note that $G(\mathbb{Q})$ is discrete and co-compact in $G(\widehat{\mathbb{Q}})$ (see, e.g., [4], [3, 5.6], [12, Chapter 5]).

In this paper, we consider the cases where the arithmetic subgroup Γ is contained in a unique maximal compact subgroup K_p of $G(\mathbb{Q}_p)$, for all primes p. We call such Γ *globally maximal*; examples are provided by finite groups Γ with globally irreducible representations V over $\mathbb{Q}$ where G is a classical group $O(V)$, $SU_F(V)$, or $SU_D(V)$, according to whether the commuting algebra of V is $\mathbb{Q}$, an imaginary quadratic field F or a definite quaternion algebra D. Other, in general not globally irreducible, examples are provided by the finite absolutely irreducible rational matrix groups that are "lattice sparse" of even type (see [9]). These are finite subgroups $\Gamma \leqslant \mathrm{GL}_n(\mathbb{Q})$ for which the natural representation is absolutely irreducible such that all Γ-invariant lattices can be obtained from any Γ-invariant lattice L, by successively taking the dual lattice, scalar multiples, intersections and sums of lattices that are already constructed (there are many such groups, e.g., for $n = 24$ there are 34 such maximal finite groups). Here the algebraic group G is $G = O(V)$ and the maximal compact subgroup $G(\mathbb{Q}_p)$ containing Γ is $O(L \otimes \mathbb{Z}_p)$ for any Γ-invariant lattice L.

Another simple example of a globally maximal Γ is the group $\Gamma = S_4 = 2^2 \rtimes \mathrm{SL}_2(2)$, which has a unique irreducible representation V of dimension 3 and determinant 1. This

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representation is orthogonal, and Γ is an arithmetic subgroup of $G = SO(V)$. The unique maximal compact K_p containing Γ is hyperspecial, for $p \neq 2$, and $K_2 = G(\mathbb{Q}_2)$. In this paper, we will consider similar examples, when G is the unique anisotropic form of G_2, F_4, and E_8 over $\mathbb{Q}$. In these cases, G is split over $\mathbb{Q}_p$ for all primes p.

In [8] the first author has already given some examples of globally maximal Γ, where K_p is hyperspecial for all primes p. These are groups over $\mathbb{Z}$, such as $\Gamma = G_2(2)$ in G of type G_2, and $\Gamma = {}^3D_4(2).3$ in G of type F_4. Here we will consider more exotic cases of (normalizers of) Jordan subgroups Γ as further examples. For these groups K_p is not hyperspecial at a single prime p. To identify the maximal parahoric subgroup K_p containing Γ at this prime, we will determine the discriminant of its Lie algebra with respect to a multiple of the Killing form.

We begin with a review of the structure of simple, simply-connected complex Lie groups $G = G(\mathbb{C})$ and their Lie algebras $\mathfrak{g}_{\mathbb{C}}$. We describe the Chevalley lattice $\mathfrak{g}$ and the associated split group G over $\mathbb{Z}$. This gives us a hyperspecial maximal compact subgroup $G(\mathbb{Z}_p)$ in $G(\mathbb{Q}_p)$ and we describe the other maximal parahoric subgroups K_p and their Lie algebras starting from $G(\mathbb{Z}_p)$. We then consider the Killing form on $\mathfrak{g}$ and show that it is divisible by $2h^\vee$, where $h^\vee$ is the dual Coxeter number. The same holds for the Lie algebras of the other maximal parahorics. We compute the discriminants of the resulting scaled forms. To give some examples for globally maximal groups, we consider the Jordan subgroups $\Gamma = 2^3.\,\mathrm{SL}_3(2) \leqslant G_2$, $3^3 \rtimes \mathrm{SL}_3(3) \leqslant F_4$, and $2^5.\,\mathrm{SL}_5(2) \leqslant 2^5.2^{10}.\,\mathrm{SL}_5(2) \leqslant E_8$ and determine the Γ-invariant lattices in $\mathfrak{g}_{\mathbb{Q}}$. The Γ-invariant Lie brackets on $\mathfrak{g}$ are unique up to scalar multiples, except for $\Gamma = 3^3 \rtimes \mathrm{SL}_3(3) \leqslant F_4$, where there are two possible Lie brackets (which are interchanged by an outer automorphism). We show that these Jordan subgroups are globally maximal and determine their maximal compact overgroups $K_p \leqslant G(\mathbb{Q}_p)$. Using an obvious generalization of the notion of globally maximal groups to arbitrary number fields, the last section shows that the Jordan subgroups of the classical groups are also globally maximal.

2. Simple Lie groups

In this section we summarize some well-known facts on Lie algebras and Lie groups. The reader can find most of the results in [13].

Let G be a simple, simply-connected, complex Lie group. Let

$$T \subset B \subset G$$

be a maximal torus contained in a Borel subgroup of G. Let $X^\bullet$ denote the character group of T. This is a free abelian group, containing the finite set Φ of roots—the non-zero characters of T which occur on $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\Phi_+ \subset \Phi$ be the positive roots, which occur on $\mathrm{Lie}(B)$, and let

$$\Delta \subset \Phi_+$$

be the root basis determined by B. Every root β in Φ_+ can be written uniquely as

$$\beta = \sum_{\alpha \in \Delta} n_\alpha(\beta)\alpha, \quad n_\alpha \geqslant 0.$$

Since G is simple, there is a highest root β_0 with the property that

$$n_\alpha(\beta_0) \geqslant 1 \quad \text{for all } \alpha \in \Delta,$$
$$n_\alpha(\beta_0) \geqslant n_\alpha(\beta) \quad \text{for all } \beta \in \Phi_+, \alpha \in \Delta.$$

The sum

$$h := 1 + \sum_{\alpha \in \Delta} n_\alpha(\beta_0)$$

is the Coxeter number of G. Let $\mathfrak{t} := \mathrm{Lie}(T)$. Then, as a representation of T,

$$\mathfrak{g} = \mathfrak{t} + \bigoplus_{\beta \in \Phi} \mathfrak{g}^\beta,$$

where each root space $\mathfrak{g}^\beta$ has dimension 1. The space $\mathfrak{t}^\beta := [\mathfrak{g}^\beta, \mathfrak{g}^{-\beta}]$ has dimension 1 and is contained in $\mathfrak{t}$. It has a unique basis H_β which satisfies $\beta(H_\beta) = 2$, here we have identified $\mathrm{Hom}(\mathfrak{t}, \mathbb{C})$ with $X^\bullet \otimes \mathbb{C}$. If X_β is a basis for $\mathfrak{g}^\beta$, there is a unique basis vector Y_β for $\mathfrak{g}^{-\beta}$ with

$$[X_\beta, Y_\beta] = H_\beta.$$

Furthermore, we have $[H_\beta, X_\beta] = 2X_\beta$, $[H_\beta, Y_\beta] = -2Y_\beta$. Hence

$$\mathfrak{g}_\beta := \langle H_\beta, X_\beta, Y_\beta \rangle$$

is a sub-algebra of $\mathfrak{g}$ isomorphic to $\mathfrak{s}l_2$.

By Lie's theorem, the homomorphism $\mathfrak{s}l_2 \to \mathfrak{g}$ given by the root β lifts to a homomorphism of complex Lie groups $\mathrm{SL}_2 \to G$. The unipotent subgroup $\mathbb{G}_a \cong \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ of SL_2 maps to the root group U_β of G, with Lie algebra $\mathfrak{g}^\beta$. The map of the tori $\mathbb{G}_m \cong \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \to T$ is the co-root $\beta^\vee$ in $X_\bullet = \mathrm{Hom}(X^\bullet, \mathbb{Z})$. Under the identification $X_\bullet \otimes \mathbb{C} = \mathrm{Lie}(T)$, $\beta^\vee$ maps to the vector H_β in $\mathfrak{t}^\beta$. Since G is simply-connected, the co-roots span $X_\bullet$, and the simple co-roots $\alpha^\vee$, $\alpha \in \Delta$ give a $\mathbb{Z}$-basis.

The Weyl group $W = N_G(T)/T$ acts on $X_\bullet$ and $X^\bullet$, and the pairing $X_\bullet \otimes X^\bullet \to \mathbb{Z}$ is W-invariant. Since G is simple, the action of W on $X_\bullet \otimes \mathbb{Q}$ is (absolutely) irreducible. Hence there is a unique W-invariant pairing

$$\langle , \rangle : X_\bullet \otimes X_\bullet \to \mathbb{Z}$$

which is even, indivisible, and positive definite. We have

$$\langle \alpha^\vee, \alpha^\vee \rangle = 2$$

if α is a long root, and

$$\langle \alpha^\vee, \alpha^\vee \rangle = 2c$$

if α is a short root (so $\alpha^\vee$ is a long co-root), with $c = 2$ or 3. The dual Coxeter number $h^\vee$ is defined by

$$h^\vee = 1 + \sum_{\alpha \text{ long}} n_\alpha(\beta_0) + \frac{1}{c} \sum_{\alpha \text{ short}} n_\alpha(\beta_0).$$

Here is a table:

G	h	c	$h^\vee$
A_n	$n+1$	1	$n+1$
B_n	$2n$	2	$2n-1$
C_n	$2n$	2	$n+1$
D_n	$2n-2$	1	$2n-2$
G_2	6	3	4
F_4	12	2	9
E_6	12	1	12
E_7	18	1	18
E_8	30	1	30

3. Integral theory

We now modify our notation slightly: the complex Lie groups and Lie algebras of the previous section will now be denoted by $G_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$ to preserve G and $\mathfrak{g}$ for the integral forms.

Chevalley proved that one can choose the basis elements X_β of the root eigenspaces $\mathfrak{g}^\beta \subset \mathfrak{g}_{\mathbb{C}}$ so that

$$[X_\beta, X_{-\beta}] = H_\beta,$$
$$[X_\beta, X_\alpha] = 0 \quad \text{if } \alpha + \beta \neq 0 \text{ is not a root},$$
$$[X_\beta, X_\alpha] = \pm(m+1)X_{\alpha+\beta} \quad \text{if } \alpha + \beta \text{ is a root}.$$

Here $m \geqslant 0$ is the largest integer such that $\beta - m\alpha$ is a root; an examination of the root systems of rank 2 shows that $m = 0, 1,$ or 2. The abelian subgroup $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$ spanned by the H_β and X_β, $\beta \in \Phi$ is a Lie order with $\mathbb{Z}$-basis $\{H_\alpha, X_\beta \mid \alpha \in \Delta,\ \beta \in \Phi\}$. This is the Lie algebra of the split, simply-connected group G over $\mathbb{Z}$ with complex points $G(\mathbb{C}) = G_{\mathbb{C}}$.

The group G is generated by the integral torus $T = X_\bullet \otimes \mathbb{G}_m$, and the root subgroups $U_\beta \cong \mathbb{G}_a$ with Lie algebras $\mathbb{Z}X_\beta$.

The group $G(\mathbb{Z}_p)$ gives a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$ for every prime p. This contains the Iwahori subgroup I_p, with reduction to the Borel B mod p. We have

$$\mathrm{Lie}(I_p) = \bigoplus_{\alpha\in\Delta} \mathbb{Z}_p H_\alpha \oplus \bigoplus_{\beta<0} \mathbb{Z}_p X_\beta \oplus \bigoplus_{\beta>0} p\mathbb{Z}_p X_\beta.$$

We want to describe the maximal parahoric subgroups of $G(\mathbb{Q}_p)$ which contain I_p. Besides $G(\mathbb{Z}_p)$, they are indexed by the simple roots α in Δ, and the groups

$$\bigl\{G(\mathbb{Z}_p),\ G_\alpha(\mathbb{Z}_p) \bigm| \alpha \in \Delta\bigr\}$$

represent the $(l+1)$ distinct conjugacy classes of maximal compact subgroups of $G(\mathbb{Q}_p)$.

To each simple root $\alpha \in \Delta$ we can associate a maximal parabolic subgroup P_α of $G(\mathbb{F}_p)$, which contains B. Its inverse image J_α in $G(\mathbb{Z}_p)$ has Lie algebra

$$\mathrm{Lie}(J_\alpha) = \bigoplus_{\gamma\in\Delta} \mathbb{Z}_p H_\gamma \oplus \bigoplus_{n_\alpha(\beta)\leqslant 0} \mathbb{Z}_p X_\beta \oplus \bigoplus_{n_\alpha(\beta)>0} p\mathbb{Z}_p X_\beta.$$

J_α is a non-maximal parahoric subgroup, and we will see that

$$J_\alpha = G(\mathbb{Z}_p) \cap G_\alpha(\mathbb{Z}_p).$$

The next theorem follows from Bruhat–Tits theory.

Theorem 1. *Let $\alpha \in \Delta$ be a simple root. Then there is a maximal compact subgroup*

$$G_\alpha := G_\alpha(\mathbb{Z}_p) \leqslant G(\mathbb{Q}_p)$$

with Lie algebra

$$\mathrm{Lie}(G_\alpha) = \mathfrak{g}_\alpha := \bigoplus_{\gamma\in\Delta} \mathbb{Z}_p H_\gamma \oplus \bigoplus_{n_\alpha(\beta)=-n} \frac{1}{p}\mathbb{Z}_p X_\beta \oplus \bigoplus_{-n<n_\alpha(\beta)\leqslant 0} \mathbb{Z}_p X_\beta \oplus \bigoplus_{n_\alpha(\beta)>0} p\mathbb{Z}_p X_\beta,$$

where $n := n_\alpha(\beta_0)$ is the multiplicity of α in the highest root β_0. The group $\overline{G_\alpha} := G_\alpha(\mathbb{F}_p)$ is a semidirect product

$$\overline{G_\alpha} = G_\alpha^{\mathrm{red}} \ltimes R(\overline{G_\alpha}),$$

where $R(\overline{G_\alpha})$ is the unipotent radical, and G_α^{red} is semi-simple, with root system

$$\Phi_\alpha = \bigl\{\beta \in \Phi \bigm| n_\alpha(\beta) \equiv 0 \pmod{n}\bigr\}.$$

This root system has simple roots $\Delta \setminus \{\alpha\} \cup \{-\beta_0\}$ with respect to the Borel subgroup reducing to I_p. The unipotent radical $R := R(\overline{G_\alpha})$ is filtered as a G_α^{red}-module, with $n-1$ abelian subquotients U_i

$$R = R_1 \supset R_2 \supset \cdots \supset R_n = \{0\}, \qquad U_i = R_i / R_{i+1} \cong \bigoplus_{n_\alpha(\beta) \equiv i \pmod n} \mathbb{F}_p X_\beta.$$

Proof. The $\mathbb{Z}_p$-lattice $\mathfrak{g}_\alpha$ is a sub-Lie order of $\mathfrak{g} \otimes \mathbb{Q}_p$. This will be the Lie algebra of G_α. Indeed, we may define G_α by adjoining to J_α the elements $e_\beta(1/p)$ in the root groups $U_\beta \otimes \mathbb{Q}_p$, where β is a root with $n_\alpha(\beta) = -n = n_\alpha(-\beta_0)$, and $e_\beta : \mathbb{G}_a \to U_\beta$ is the isomorphism over $\mathbb{Z}_p$. This gives a compact subgroup with desired Lie algebra, by the Chevalley relations. The theory of Bruhat and Tits shows that G_α defines a smooth group scheme over $\mathbb{Z}_p$, and describes its special fiber. The filtration of R is obtained by looking at the orbits of the Weyl group of G_α^{red} on Φ. $\square$

Remark 2. When G is simply-laced, each U_i is a minuscule, irreducible representation of G_α^{red}. In general, there are at most two orbits of the Weyl group of G_α^{red} on the weights in U_i, corresponding to the roots β of different lengths with $n_\alpha(\beta) \equiv i \pmod n$. In this case, U_i need not be irreducible, if $p = c$.

Remark 3. The semi-direct product structure of $\overline{G_\alpha}$ gives a Lie ideal M with

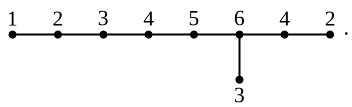
$$L = \mathrm{Lie}(G_\alpha) \supset M \supset pL.$$

The quotient L/M is isomorphic to the Lie algebra of G_α^{red} and M/pL has order $p^{\dim(R(\overline{G_\alpha}))}$.

4. An example—the maximal parahorics in E_8

We illustrate the theory of the previous section with a discussion of the 9 conjugacy classes of maximal parahoric subgroups of E_8 over $\mathbb{Q}_p$. For each, we determine G_α^{red}, as well as the minuscule representations in the filtration of $R(\overline{G_\alpha})$. The representations U_i are explicitly identified using the description of their roots in Theorem 1 with the help of the system LIE [19].

The distinct conjugacy classes of maximal parahoric subgroups of $E_8(\mathbb{Q}_p)$ correspond bijectively to the nodes of the extended Dynkin diagram:



We have labelled the nodes with the multiplicity $n_\alpha(\beta_0)$ of the corresponding simple root α in the highest root β_0. The extended vertex, with label $n = 1$, corresponds to the longest root $-\beta_0$.

We discuss the parahorics from left to right. $\mu_a \leqslant G_m$ denotes the group of ath roots of unity. By diag we understand a diagonal embedding, which is determined by the representation occurring in $R(\overline{G_\alpha})$.

- The unique vertex labelled 1 corresponds to the hyperspecial compact $G(\mathbb{Z}_p)$. This has

$$G^{\text{red}} = E_8, \qquad R(\overline{G}) = 0, \qquad \dim\bigl(R(\overline{G})\bigr) = 0.$$

- The adjacent vertex, labelled 2, has

$$G_\alpha^{\text{red}} \cong (\mathrm{SL}_2 \times E_7)/\operatorname{diag}(\mu_2), \qquad R(\overline{G_\alpha}) = U_1 = 2 \otimes 56, \qquad \dim\bigl(R(\overline{G_\alpha})\bigr) = 112,$$

  where we have indicated a minuscule representation of a factor by its dimension.
- The adjacent vertex, labelled 3, has

$$G_\alpha^{\text{red}} \cong (\mathrm{SL}_3 \times E_6)/\operatorname{diag}(\mu_3),$$
$$U_1 = 3 \otimes 27, \qquad U_2 = 3' \otimes 27',$$
$$\dim\bigl(R(\overline{G_\alpha})\bigr) = 162.$$

  Here $3'$ is the contragredient representation of the natural representation 3 of SL_3, and the representations 27 and $27'$ of E_6 are also dual.
- The adjacent vertex, labelled 4, has

$$G_\alpha^{\text{red}} \cong (\mathrm{SL}_4 \times \mathrm{Spin}_{10})/\operatorname{diag}(\mu_4),$$
$$U_1 = 4 \otimes 16, \qquad U_2 = 6 \otimes 10, \qquad U_3 = 4' \otimes 16',$$
$$\dim\bigl(R(\overline{G_\alpha})\bigr) = 188.$$

  Here 4 is the natural representation of SL_4, $6 = \bigwedge^2(4)$, and $4' = \bigwedge^3(4)$ is the dual of 4. The representations 16 and $16'$ are the half spin representations of $\mathrm{Spin}(10)$.
- The adjacent vertex, labelled 5, has

$$G_\alpha^{\text{red}} \cong (\mathrm{SL}_5 \times \mathrm{SL}_5)/\operatorname{diag}(\mu_5),$$
$$U_1 = 5 \otimes 10, \qquad U_2 = 10 \otimes 5', \qquad U_3 = 10' \otimes 5, \qquad U_4 = 5' \otimes 10',$$
$$\dim\bigl(R(\overline{G_\alpha})\bigr) = 200.$$

  Here 5 is the natural representation of SL_5, $10 = \bigwedge^2 5$, $10' = \bigwedge^3 5$, and $5' = \bigwedge^4 5$.
- The adjacent vertex, labelled 6, has

$$G_\alpha^{\text{red}} \cong (\mathrm{SL}_2 \times \mathrm{SL}_3 \times \mathrm{SL}_6)/\operatorname{diag}(\mu_6),$$
$$U_1 = 2 \otimes 3 \otimes 6, \qquad U_2 = 1 \otimes 3' \otimes 15, \qquad U_3 = 2 \otimes 1 \otimes 20,$$
$$U_4 = 1 \otimes 3 \otimes 15', \qquad U_5 = 2 \otimes 3' \otimes 6',$$
$$\dim\bigl(R(\overline{G_\alpha})\bigr) = 202.$$

Here 6 is the natural representation of SL_6, $15 = \bigwedge^2 6$, $20 = \bigwedge^3 6$, $15' = \bigwedge^4 6$, and $6' = \bigwedge^5(6)$.

- The bottom vertex, adjacent with 6 and labelled 3, has

$$G_\alpha^{\mathrm{red}} \cong \mathrm{SL}_9/\mu_3,$$
$$U_1 = 84 = \bigwedge^3 9, \qquad U_2 = 84' = \bigwedge^6 9,$$
$$\dim\bigl(R(\overline{G_\alpha})\bigr) = 168,$$

where 9 is the natural representation of SL_9.

- The next vertex, adjacent to 6 and labelled 4, has

$$G_\alpha^{\mathrm{red}} \cong (\mathrm{SL}_2 \times \mathrm{SL}_8)/\operatorname{diag}(\mu_2),$$
$$U_1 = 2 \otimes 28, \qquad U_2 = 1 \otimes 70, \qquad U_3 = 2 \otimes 28',$$
$$\dim\bigl(R(\overline{G_\alpha})\bigr) = 182.$$

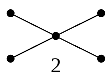
Here 8 is the natural representation of SL_8, $28 = \bigwedge^2 8$, $70 = \bigwedge^4 8$, and $28' = \bigwedge^6 8$. Similarly 2 is the natural representation of SL_2 (which is self-dual) and 1 is the trivial representation of SL_2.

- The last vertex on the right, labelled 2, has

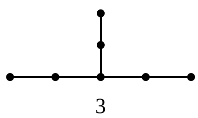
$$G_\alpha^{\mathrm{red}} \cong \mathrm{Spin}_{16}/\mu_2, \qquad R(\overline{G_\alpha}) = U_1 = 128, \qquad \dim\bigl(R(\overline{G_\alpha})\bigr) = 128.$$

In each case it is interesting to note that every minuscule representation of G_α^{red} occurs in the filtration of $R(\overline{G_\alpha})$. This is a general phenomenon, when G is of adjoint type, as the center of G_α^{red} has order $n = n_\alpha(\beta_0)$.

Some other examples of maximal parahorics, which exhibit unusual symmetry, are given by the following simple roots α, indicated in the extended Dynkin diagram:

- $G = \mathrm{Spin}_8$: 

$$G_\alpha^{\mathrm{red}} \cong (\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2)/\operatorname{diag}(\mu_2),$$
$$R(\overline{G_\alpha}) = U_1 = 2 \otimes 2 \otimes 2 \otimes 2, \qquad \dim\bigl(R(\overline{G_\alpha})\bigr) = 16;$$

- $G = E_6$: 

$$G_\alpha^{\mathrm{red}} \cong (\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{SL}_3)/\operatorname{diag}(\mu_3),$$
$$U_1 = 3 \otimes 3 \otimes 3, \qquad U_2 = 3' \otimes 3' \otimes 3',$$
$$\dim\bigl(R(\overline{G_\alpha})\bigr) = 54.$$

5. The Killing form

We retain the notion of Section 3, so $\mathfrak{g} = \mathrm{Lie}(G)$ is the Chevalley Lie algebra of the simply-connected, simple group scheme G over $\mathbb{Z}$. The Killing form

$$(X, Y) := \mathrm{Tr}(\mathrm{ad}\, X \cdot \mathrm{ad}\, Y)$$

is integral, symmetric, and G-invariant on $\mathfrak{g}$. On $X_\bullet(T) = \mathrm{Lie}(T)$, it is integral, even, and W-invariant, so it is a multiple of the indivisible form $\langle,\rangle$ with $\langle \alpha^\vee, \alpha^\vee \rangle = 2$ for α a long root. Steinberg and Springer [14] show that

$$(H_\alpha, H_\alpha) = 4h^\vee$$

for α a long root, with $h^\vee$ the dual Coxeter number. Hence

$$(,) = 2h^\vee \cdot \langle,\rangle$$

as bilinear forms on $\mathrm{Lie}(T)$. The decomposition

$$\mathrm{Lie}(G) = \mathrm{Lie}(T) \oplus \bigoplus_{\beta>0} (\mathbb{Z}X_\beta + \mathbb{Z}X_{-\beta})$$

is orthogonal for the Killing form. Steinberg and Springer also show that, for all roots β,

$$(X_\beta, X_{-\beta}) = \frac{1}{2}(H_\beta, H_\beta) = h^\vee \langle H_\beta, H_\beta \rangle.$$

Hence, if we define

$$\langle X, Y \rangle := \frac{1}{2h^\vee}(X, Y),$$

we find that

Proposition 4. *The pairing*

$$\langle,\rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{Z}$$

is even, indivisible, and is positive definite on $\mathrm{Lie}(T)$.

If G is simply-laced, we find that, with respect to $\langle,\rangle$,

$$\mathfrak{g}^*/\mathfrak{g} \cong \mathrm{Lie}(T)^*/\,\mathrm{Lie}(T) \cong \widehat{Z(G)},$$

where $Z(G)$ is the (finite) center of G. In the general case, $\mathfrak{g}^*/\mathfrak{g}$ has order $\#Z(G)c^k$, where c is given in the table in Section 2 and k is the number of short positive roots plus the number of short simple roots.

Here is a table:

G	$\det\langle,\rangle$ on $\mathfrak{g}$	$\det\langle,\rangle$ on $\mathrm{Lie}(T)$
A_n	$n+1$	$n+1$
B_n	2^{n+2}	2^2
C_n	2^{n^2}	2^n
D_n	2^2	2^2
G_2	3^7	3
F_4	2^{26}	2^2
E_6	3	3
E_7	2	2
E_8	1	1

The pairing $\langle,\rangle$ on $\mathfrak{g}\otimes\mathbb{Q}_p$ is also integral and even on the $\mathbb{Z}_p$-lattices $L=\mathrm{Lie}(G_\alpha)$, for the maximal parahorics in $G(\mathbb{Q}_p)$ defined in Section 3. Indeed, the only change in the discriminant L^*/L from that of $\mathfrak{g}^*/\mathfrak{g}$ involves the planes

$$\mathbb{Z}_p X_{-\beta} + p\mathbb{Z}_p X_\beta,$$

where β is a positive root with

$$0 < n_\alpha(\beta) < n_\alpha(\beta_0).$$

This contributes a factor of $(\mathbb{Z}/p\mathbb{Z})^2$ to L^*/L. Hence, we find the following proposition.

Proposition 5. *Assume that p does not divide* $\det(\langle,\rangle)$ *on* $\mathfrak{g}$. *Then*

$$L^*/L \cong (\mathbb{Z}/p\mathbb{Z})^{\dim R(\overline{G_\alpha})}$$

and pL^* is the Lie ideal M with $L/M \cong$ $\mathrm{Lie}(G_\alpha^{\mathrm{red}})$.

This allows us to determine which maximal parahorics $G_\alpha(\mathbb{Z}_p)$ can contain certain finite groups $\Gamma\subset G(\mathbb{Q}_p)$, once we know some information on the Γ-stable lattices in $\mathfrak{g}\otimes\mathbb{Q}_p$.

6. The type of some Jordan subgroups of exceptional groups

Definition 6. Let Γ be an arithmetic subgroup of a linear algebraic group G defined over some number field F. Then Γ is called *globally maximal* if for all prime ideals $\wp$ of F there is a unique maximal compact subgroup $\mathcal{T}_\wp(\Gamma)$ of $G(F_\wp)$ over the $\wp$-adic completion $F_\wp$ of F, that contains Γ. In this case the *type* of Γ is

$$\mathcal{T}(\Gamma) := \big(\mathcal{T}_\wp(\Gamma)\big)_{\wp \text{ prime}}.$$

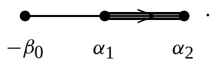
If Γ is a globally maximal group, then the type $\mathcal{T}_\wp(\Gamma)$ is hyperspecial for almost all primes $\wp$. In particular, if the Γ-module $\mathrm{Lie}(G(F_\wp))$ is irreducible modulo $\wp$ then $\mathcal{T}_\wp(\Gamma)$ is a hyperspecial.

In abuse of notation, we call a finite subgroup N of a complex simple Lie group G a *Jordan subgroup*, if $N = N_G(J)$ for some elementary abelian subgroup J of G which is a minimal normal subgroup in N, if N satisfies the following maximality condition: for any abelian subgroup $\widetilde{J} \supset J$ such that $N_G(\widetilde{J}) \supset N$ one has $N = N_G(\widetilde{J})$. Usually the minimal normal subgroup J is called a Jordan subgroup. The Jordan subgroups have been classified by Alekseevskii [1] (see also [15, p. 505]). The minimal splitting fields for the Jordan subgroups have been determined in [6].

We now treat the different Jordan subgroups of the exceptional groups in detail. The two missing Jordan subgroups $3^{3+3}.\mathrm{SL}_3(3) \leqslant E_6$ and $5^3.\mathrm{SL}_3(5) \leqslant E_8$ are other interesting groups which could be considered. The explicit calculations below are performed using the computer algebra system MAGMA [18].

6.1. $2^3.\mathrm{SL}_3(2)$ *in* G_2

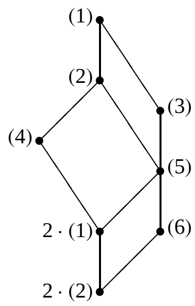
The simple roots of G_2 are given as follows:



The next theorem is already shown in [5] by calculations in the 7-dimensional representation of G_2.

Theorem 7. *Let $\Gamma := 2^3.\mathrm{SL}_3(2)$ be the Jordan subgroup of the anisotropic form $G(\mathbb{Q})$ of G_2. Then $\mathcal{T}_p(\Gamma) = G_2$ is hyperspecial for $p > 2$ and $\mathcal{T}_2(\Gamma) = A_2$.*

Proof. Γ has a unique complex irreducible 14-dimensional representation V. This representation is rational. The space of Γ-invariant homomorphisms of $V \otimes V$ to V is one-dimensional. Any generator of this space is skew symmetric and gives a Γ-invariant Lie multiplication on V. This yields an embedding of Γ into $G(\mathbb{Q})$. The group Γ fixes up to isomorphism 12 lattices in V of which the 2-local inclusions are given as follows:



Here the vertical line (e.g., from (1) to (2)) and the lines parallel to (2) (4) indicate inclusions of index 2^3 (two different $\mathbb{F}_2\Gamma$-modules) and the lines parallel to (1) (3) mean inclusions of index 2^8. The other 6 isomorphism classes are represented by sublattices of index 3^7 of these 6 lattices. This gives the type of Γ for all primes $p > 2$. The $G_{\alpha_1}(\mathbb{Z}_2)$-composition factors of $\mathrm{Lie}(G_{\alpha_1}(\mathbb{Z}_2))/2\,\mathrm{Lie}(G_{\alpha_1}(\mathbb{Z}_2))$ are of dimension 1, 2, and 4. Hence the 2-local type of Γ is either G_2 or A_2. It follows from the mass-formula (see [5,8]) or from the calculation in [5] that $\mathcal{T}_2(\Gamma) = G_{\alpha_2}(\mathbb{Z}_2)$ is of type A_2. $\square$

Remark 8. The reduction map $\Gamma \to G_{\alpha_2}(\mathbb{F}_2)$ is injective.

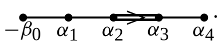
Remark 9. The possibility that $\mathcal{T}_2(\Gamma) = G_2$ cannot be ruled out looking at the Lie bracket: the maximal Γ-invariant Lie order (which corresponds to the lattice (1) in the picture above) has discriminant 3^7 (with respect to $1/8$ times the Killing form) which is the same discriminant as the one of $\mathrm{Lie}(G(\mathbb{Z}))$. Indeed, this Lie order is also invariant under the maximal compact $G_{\alpha_2}(\mathbb{Z}_2)$ of type A_2 that contains Γ. The Lie order $\mathrm{Lie}(G_{\alpha_2}(\mathbb{Z}_2))$ corresponds to the lattice number (2) in the picture above, which is contained in (1) of index 2^3.

6.2. $3^3 \rtimes \mathrm{SL}_3(3)$ *in* F_4

Let Γ be the Jordan subgroup $3^3 \rtimes \mathrm{SL}_3(3)$ of the unique anisotropic form $G(\mathbb{Q})$ of the algebraic group F_4.

Explicit character calculations show that Γ has 3 absolutely irreducible representations of degree 52. To decide which one is the action of Γ on the Lie algebra of $G(\mathbb{Q})$, we note that the elements of order 9 in both conjugacy classes of $G(\mathbb{Q})$ have trace 1 in the adjoint representation (see [11]). This identifies the representation $V = \mathrm{Lie}(G(\mathbb{Q}))$ of Γ uniquely. The space $H := \mathrm{Hom}_\Gamma(\bigwedge^2 V, V)$ is 2-dimensional. Explicit construction of this representation V shows that the Jacobi identity gives a quadratic equation which has two solutions in H which can be calculated with [18]. Hence there are up to scalar multiples two Γ-invariant Lie brackets on V. They are interchanged by the outer automorphism (in $3^3 \rtimes \mathrm{GL}_3(3)$) of Γ (which is not in $G(\mathbb{Q})$), therefore there are up to conjugacy two representations of Γ into $G(\mathbb{Q})$ giving the same conjugacy class of groups $\Gamma \leqslant G(\mathbb{Q})$. We fix one of the two Γ-invariant Lie brackets.

The simple roots of F_4 are indicated in the following diagram:



Theorem 10. *$\mathcal{T}_p(\Gamma) = G(\mathbb{Z}_p)$ is hyperspecial for $p \neq 3$ and $\mathcal{T}_3(\Gamma) = G_{\alpha_2}(\mathbb{Z}_3)$.*

Proof. For $p > 3$, the theorem follows because the representation of Γ on the Lie algebra is irreducible modulo p. For $p = 2$, this representation has two 2-modular constituents of degree 26. This implies that $\mathcal{T}_2(\Gamma) = G(\mathbb{Z}_2)$ is also hyperspecial. It remains to consider the prime 3. The unique maximal Γ-invariant Lie order has discriminant $2^{26} \cdot 3^{36}$ (with respect to $1/18$ times the Killing form) which is the discriminant of the Lie order $\mathrm{Lie}(G_{\alpha_2}(\mathbb{Z}_3))$. There is no other Γ-invariant Lie order that has the discriminant of the Lie algebra of a maximal compact subgroup of $G(\mathbb{Q}_3)$. Therefore $\mathcal{T}_3(\Gamma) = G_{\alpha_2}(\mathbb{Z}_3)$. $\square$

6.3. $2^5.\,\mathrm{SL}_5(2)$ *and* $2^5.2^{10}.\,\mathrm{SL}_5(2)$ *in* E_8

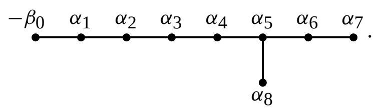
Let $\Gamma := 2^5.\,\mathrm{SL}_5(2)$ be a Jordan subgroup of the unique anisotropic form $G(\mathbb{Q})$ of the algebraic group E_8 and let $H := 2^5.2^{10}.\,\mathrm{SL}_5(2)$ be the maximal finite Jordan subgroup of $G(\mathbb{Q})$ that contains Γ.

The 248-dimensional representation V of Γ can be obtained from the 248-dimensional integral representation of the Thompson group, which contains Γ as a maximal subgroup, from the matrices in the internet atlas [17]. To construct the Γ-invariant Lie multiplication on V (which is unique up to scalar multiples) we decompose V as the direct sum of eigenspaces

$$V = \bigoplus_{\chi} V_\chi$$

under the maximal normal 2-subgroup $T \cong 2^5$ of Γ. All 31 nontrivial characters χ of T occur on V with multiplicity 8 and Γ permutes the V_χ 2-transitively. The space of $\mathrm{Stab}_\Gamma(\chi_1, \chi_2)$-invariant homomorphisms from $V_{\chi_1} \otimes V_{\chi_2}$ to $V_{\chi_1\chi_2}$ is one-dimensional. From this one constructs the Γ-invariant Lie bracket on V. The maximal decomposable sublattice $L_{OD} := \bigoplus_\chi (V_\chi \cap \Lambda)$ of the Thompson–Smith lattice Λ carries a Γ-invariant integral Lie multiplication such that the discriminant of L_{OD} (with respect to $1/60$ times the Killing form) is 2^{248}.

The simple roots of G are labelled as in the following extended Dynkin diagram:



Theorem 11. *Γ (and hence also H) is a globally maximal subgroup of E_8. The type of Γ is $\mathcal{T}(\Gamma) = \mathcal{T}(H)$ with $\mathcal{T}_2(\Gamma) = G_{\alpha_4}(\mathbb{Z}_2)$ and $\mathcal{T}_p(\Gamma) = G(\mathbb{Z}_p)$ for all primes $p > 2$.*

Proof. The 248-dimensional representation of Γ is absolutely irreducible modulo every prime $p > 2$. Therefore $\mathcal{T}_p(\Gamma) = \mathcal{T}_p(H) = G(\mathbb{Z}_p)$ for all primes $p > 2$. Since Γ is compact, it embeds into at least one maximal compact subgroup G_α of $G(\mathbb{Q}_2)$. Then Γ acts on $\mathrm{Lie}(G_\alpha)$ and hence there is a Γ-invariant Lie order in V, of the correct discriminant (with respect to $1/60$ times the Killing form) $2^{\dim(R(\overline{G_\alpha}))}$ (see Section 4). With MAGMA [18] one calculates that Γ fixes up to isomorphism 383 lattices in V. The only lattice of one of the discriminants above, that is closed under the Lie bracket is a lattice L_{A4+A4} of discriminant 2^{200}. Hence $\alpha = \alpha_4$, $\Gamma \leqslant G_{\alpha_4}(\mathbb{Z}_2)$, and $\mathcal{T}_2(\Gamma) = G_{\alpha_4}(\mathbb{Z}_2)$. $\square$

Remark 12. The representation of Γ on $L_{A_4+A_4}/2L_{A_4+A_4}$ and hence also the reduction map of Γ to $G_{\alpha_4}(\mathbb{F}_2)$ is injective. From the action of Γ on $\mathrm{Lie}(G^{\mathrm{red}}_{\alpha_4})$ one sees that the image is diagonally embedded in $G^{\mathrm{red}}_{\alpha_4} \cong \mathrm{SL}_5(2) \times \mathrm{SL}_5(2)/\operatorname{diag}\mu_5$.

Remark 13. There are two maximal Γ-invariant lattices that are closed under the Lie bracket, one of which is the orthogonal decomposition $L_{OD} \cong {}^{(2)}E_8^{31}$ of discriminant 2^{248}

and the other lattice is L_{A4+A4}. Therefore, Γ has 2 maximal Lie orders. Since both of them are also stable under H, the same holds for H. For both Lie orders, the Lie bracket is surjective. The intersection $L_{A4+A4} \cap L_{OD}$ is of index 2^5 in L_{OD} (and of index 2^{29} in L_{A4+A4}).

7. The Jordan subgroups of the classical groups

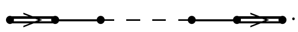
In this section we calculate the type of the Jordan subgroups of the classical groups G. To this aim we calculate in the natural representation of G. Then the group Γ may contain a center, that acts trivially in the adjoint representation of G.

The symbol $\rtimes$ denotes a split extension and a . indicates an extension that is usually non split.

7.1. $r_+^{1+2n}.Sp_{2n}(r) \leqslant U_{r^n}$

Let $r \in \mathbb{N}$ be a prime. Let $\Gamma := r_+^{1+2n} \rtimes Sp_{2n}(r) \leqslant U_{r^n}(\mathbb{C})$ if $r > 2$ and if $r = 2$ then let Γ be the central product $\Gamma = (C_4 \mathsf{Y}\, 2^{1+2n}).Sp_{2n}(2) \leqslant U_{2^n}(\mathbb{C})$. The minimal number field $\widetilde{F}$, such that Γ is contained in $U_{r^n}(\widetilde{F})$ is $\widetilde{F} = \mathbb{Q}[\zeta_8]$ for $r = 2$ and $\widetilde{F} = \mathbb{Q}[\zeta_r]$ for $r > 2$ and the involution is complex conjugation. Let F denote the totally real subfield of $\widetilde{F}$. Then the algebraic group G is defined over F.

Theorem 14. *Γ is a globally maximal subgroup of $G(\widetilde{F})$.*
(a) *For $r = 2$ the type of Γ is hyperspecial for all primes $\wp$ of F.*
(b) *For $r > 2$ the type of Γ is hyperspecial for all primes of F not dividing r. Let $q := (r^n - 1)/2$. Then at the prime $\wp := (1 - \zeta_r)(1 - \zeta_r^{-1})$, the type $\mathcal{T}_\wp(\Gamma)$ is the maximal parahoric subgroup corresponding to the vertex number $\lfloor q/2 \rfloor + 1$ of the local Dynkin diagram $C - BC_q$ (with $q + 1$ vertices)* [16, 7*th diagram on p.* 60]:



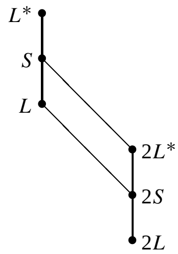
Proof. (a) For $r = 2$ the group $\langle \zeta_8 \rangle \mathsf{Y}\, \Gamma$ is the complex Clifford group described in [10, Section 6]. It follows from the remarks before [10, Theorem 6.5], that the natural representation of Γ is irreducible modulo all primes $\wp$ of F. Hence, the type of Γ is hyperspecial everywhere.

(b) Since $O_r(\Gamma)$ acts absolutely irreducible, the natural representation of Γ is clearly irreducible modulo all primes of F that do not divide r. [2] shows that the natural representation of Γ has two r-modular constituents of degree q and $q + 1$, where $r^n = 2q + 1$. Hence, Γ embeds into the maximal parahoric subgroup P of $U_{r^n}(F_\wp)$ with $P^{\mathrm{red}} = O_q(r) \times Sp_{q+1}(r)$ if q is odd and $P^{\mathrm{red}} = O_{q+1}(r) \times Sp_q(r)$ if q is even. $\square$

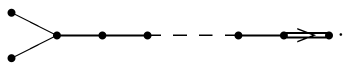
Note that also for the case $r > 2$, the group Γ is (up to certain scalars) the Clifford group described in [10, Section 7]. In particular Γ is (up to scalars) a maximal finite subgroup of $U_{r^n}(\mathbb{C})$ (see [10, Theorem 7.3] for $r > 2$ and [10, Theorem 6.5] for $r = 2$).

7.2. $2^{2n} \rtimes S_{2n+1} \leqslant O_{2n+1}$

Let $n \geqslant 3$ and $\Gamma := 2^{2n} \rtimes S_{2n+1}$. Then Γ is the determinant 1 subgroup of the full monomial subgroup $M \leqslant O_{2n+1}(\mathbb{Q})$. M is generated by $-I_{2n+1}$ and Γ. Hence Γ fixes the same lattices as M, namely the standard lattice $S := \mathbb{Z}^{2n+1}$ with quadratic form $f := \sum_{i=1}^{2n+1} x_i^2$, its even sublattice L and the dual lattice L^* (see, e.g., [7]).



The unique maximal parahoric subgroup P of $O_{2n+1}(\mathbb{Q}_2, f)$ that contains Γ is the orthogonal group of the lattice $L \otimes \mathbb{Z}_2$. It also stabilizes the dual lattice L^* and, since $L^*/L \cong C_4$, the unique lattice S between L^* and L. P corresponds to the last vertex of the local Dynkin diagram

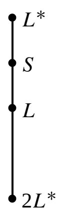


Note that $\overline{P}$ is isomorphic to $2^{2n} \rtimes O^+_{2n}(2)$, if $2n+1 \equiv \pm 3 \pmod 8$, i.e., the 2-adic quadratic space is split over $\mathbb{Q}_2$ [16, Dynkin diagram B_n, p. 60] and $\overline{P} \cong 2^{2n} \rtimes O^-_{2n}(2)$, if $2n+1 \equiv \pm 1 \pmod 8$, i.e., the 2-adic quadratic space is non-split over $\mathbb{Q}_2$ [16, Dynkin diagram ${}^2B_n$, p. 63].

Theorem 15. *Let $\Gamma := 2^{2n} \rtimes S_{2n+1} \leqslant O_{2n+1}(\mathbb{Q})$. Then Γ is globally maximal. $\mathcal{T}_p(\Gamma)$ is hyperspecial for all primes $p > 2$ and $\mathcal{T}_2(\Gamma) = P$ as described above.*

7.3. $2^{2n-1} \rtimes S_{2n} \leqslant O_{2n}$

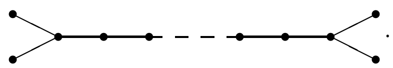
Assume that $n \geqslant 5$ and let $\Gamma := 2^{2n-1} \rtimes S_{2n}$ be the determinant 1 subgroup of the full monomial subgroup M of $O_{2n}(\mathbb{Q})$. As in the last section M fixes 3 lattices, the standard lattice $S := \mathbb{Z}^{2n}$, its even sublattice L and the dual lattice L^* (see, e.g., [7]). Since the dimension is even, $L^*/L \cong C_2 \times C_2$ and $2L^*$ is contained in L.



Since Γ is a normal subgroup of index 2 in M, the only other lattices possibly fixed by Γ are the two lattices $S_1 = \langle L, (1/2)\sum_{i=1}^{2n} x_i\rangle$ and $S_2 = \langle L, (1/2)\sum_{i=1}^{2n} x_i - x_1\rangle$. These are not fixed under Γ (the stabilizer in M of either of these two lattices is the subgroup of M generated by all permutations and all even sign changes) and hence Γ fixes the same lattices as M.

The type of Γ is clearly hyperspecial for all primes $p > 2$. For $p = 2$, the type $P := \mathcal{T}_2(\Gamma)$ is as follows.

Assume first that $2n \equiv 0 \pmod 8$, i.e., the 2-adic quadratic space is split. In this case the local Dynkin diagram is D_n on [16, p. 61]:



The two lattices S_1 and S_2 are even unimodular lattices and their 2-adic stabilizers correspond to the two extremal hyperspecial vertices at one side of the diagram above. Γ interchanges the two lattices S_1 and S_2 and hence fixes the midpoint m of the edge joining the two hyperspecial maximal parahorics in the building. The type of Γ is the maximal compact group $P = \mathrm{Stab}(m)$.

Now assume that $2n \not\equiv 0 \pmod 8$. Then the dimension of the anisotropic kernel of the quadratic space is 4 and the relative local Dynkin diagram is ${}^2D_n'$ on [16, p. 65]:

In this case, P corresponds to one of the two (special) extremal vertices labelled by 2.

Theorem 16. *Let $\Gamma := 2^{2n-1} \rtimes S_{2n} \leqslant SO_{2n}(\mathbb{Q})$. Then Γ is globally maximal. $\mathcal{T}_p(\Gamma)$ is hyperspecial for $p > 2$. For $p = 2$, $\mathcal{T}_2(\Gamma) = P$ as described above.*

7.4. $2_+^{1+2n}.O_{2n}^+(2) \leqslant O_{2^n}$

Let $n \geqslant 3$ and $\Gamma := 2_+^{1+2n}.O_{2n}^+(2) \leqslant O_{2^n}(\mathbb{R})$. Then Γ is the full normalizer of the extraspecial group 2_+^{1+2n} in the orthogonal group $O_{2^n}(\mathbb{R})$. This gives an isomorphism of Γ with the real Clifford group described in [10]. Hence up to conjugacy $\Gamma \leqslant O_{2^n}(F)$, with $F := \mathbb{Q}[\sqrt{2}]$ and by [10, Lemma 5.4] Γ fixes only one lattice in F^{2^n}. Therefore, Γ is globally maximal and the type of Γ is hyperspecial for all primes $\wp$ of $\mathbb{Z}[\sqrt{2}]$. The reduction modulo $\sqrt{2}$ of Γ is the natural action of $O_{2n}^+(2)$ on the simple module of the Clifford algebra of the associated quadratic form.

Theorem 17. *Let $\Gamma := 2_+^{1+2n}.O_{2n}^+(2) \leqslant O_{2^n}(F)$, where $F = \mathbb{Q}[\sqrt{2}]$. Then Γ is globally maximal and the type of is hyperspecial for all primes $\wp$ of F.*

Note that Γ is a maximal finite subgroup of $O_{2^n}(\mathbb{R})$ as shown in [10, Theorem 5.6].

7.5. $2_-^{1+2n}.O_{2n}^-(2) \leqslant Sp_{2^n}$

The group $\Gamma := 2_-^{1+2n}.O_{2n}^-(2)$ is the centralizer of one factor Q_8 in $2_+^{1+2(n+1)}.O_{2n+2}^+(2)$. Let $\mathcal{Q}$ be the quaternion algebra with center $F := \mathbb{Q}[\sqrt{2}\,]$ ramified only at the two infinite places. Then Γ can be realized as a subgroup of $U_{2^{n-1}}(\mathcal{Q}) \leqslant Sp_{2^n}(\mathbb{C})$.

Lemma 18. *The enveloping order $\mathbb{Z}\Gamma$ of Γ in $\mathcal{Q}^{2^{n-1}\times 2^{n-1}}$ is a maximal order.*

Proof. For $n \leqslant 2$ the lemma can be checked easily by direct computations. Assume that $n \geqslant 3$. Then Γ contains the tensor product $\widetilde{S}_4 \curlyvee 2_+^{1+2(n-1)}.O_{2(n-1)}^+(2)$. Since $n-1 \geqslant 2$, the group $2_+^{1+2(n-1)}.O_{2(n-1)}^+(2) \leqslant O_{2^{n-1}}(\mathbb{Q}[\sqrt{2}])$ spans a maximal order $\cong \mathbb{Z}[\sqrt{2}]^{2^{n-1}\times 2^{n-1}}$ by [10, Lemma 5.4]. Since $\widetilde{S}_4$ spans a maximal order in $\mathcal{Q}$, the lemma follows by taking tensor products. $\square$

Hence Γ fixes only one class of lattices and hence we get the following theorem.

Theorem 19. *Let $\Gamma := 2_-^{1+2n}.O_{2n}^-(2) \leqslant U_{2^{n-1}}(\mathcal{Q}) \leqslant Sp_{2^n}(\mathbb{C})$. Then Γ is a globally maximal subgroup of $U_{2^{n-1}}(\mathcal{Q})$ and the type of Γ is hyperspecial for all primes $\wp$ of $F = \mathbb{Q}[\sqrt{2}]$.*

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